B. Math Exam

Question 1

(i) Using $\int f(x,y)dxdy = 1$ and the expression for f, we have

$$1 = \int_0^\infty \int_0^y Ce^{-\lambda x}e^{-\lambda y}dxdy = \int_0^\infty \frac{C}{\lambda}e^{-\lambda y}(1-e^{-\lambda y})dy = \frac{C}{2\lambda^2}.$$

Solving, we have $C = 2\lambda^2$.

(ii) For $x > 0$, we have

$$f_X(x) = \int f(x,y)dy = \int_x^\infty Ce^{-\lambda x}e^{-\lambda y}dy = \frac{C}{\lambda}e^{-2\lambda x}.$$

Since $C = 2\lambda^2$, we have $f_X(x) = 0$ if $x \leq 0$ and $f_X(x) = 2\lambda e^{-2\lambda x}$ for $x > 0$.

Similarly, for $y > 0$, we have

$$f_Y(y) = \int f(x,y)dx = \int_0^y Ce^{-\lambda x}e^{-\lambda y}dy = 2\lambda e^{-\lambda y}(1-e^{-\lambda y}).$$

Since $f(x,y) \neq f_X(x)f_Y(y)$ for $x, y > 0$, we have that the random variables are not independent.

Question 2

Let $f_Z(z)$ denote the density of Z. Using the convolution formula, we have

$$f_Z(z) = \int_0^z f_Y(t)f_X(z-t)dt.$$

We note that $f_Y(t) = 1$ if $0 \leq t \leq 1$ and $= 0$ else. So if $z > 1$, we have

$$f_Z(z) = \int_0^1 e^{-\lambda z}e^{\lambda t}dt = \lambda^{-1}e^{-\lambda z}(e^\lambda - 1).$$

If $z \leq 1$, we have

$$f_Z(z) = \int_0^z e^{-\lambda z} e^{\lambda t} dt = \lambda^{-1}(1 - e^{-\lambda z}).$$

Question 3

Since X and Y are independent normal and W and Z are obtained through a linear transformation, the vector (W, Z) is also jointly normal. The determinant J of the Jacobian of the transformation matrix is -3. Using $X = W/3$ and $Y = -W/3 + Z$, we have (Z, W) has a density given by

$$f_{ZW}(z, w) = |J|^{-1} f_{XY}(w/3, -w/3 + z)$$

where $f_{XY}(x, y) = (2\pi)^{-1} e^{-x^2 - y^2}$.

Question 4

(i) Since $|xy| \leq \frac{1}{2}(x^2 + y^2)$, we have

$$\int |xy| f(x, y) dx dy \leq \int \frac{1}{2}(x^2+y^2) f(x, y) dx dy = \frac{1}{2} \int x^2 f_X(x) dx + \frac{1}{2} \int y^2 f_Y(y) dy$$

which is finite. Therefore,

$$|E(XY)| = |\int xy f(x, y) dx dy| \leq \int |xy| f(x, y) dx dy$$

is finite.

(ii) We have

$$var(X_1 + ... + X_n) = E(X_1 + ... + X_n)^2 - (E(X_1 + ... + X_n))^2.$$

Using $(x_1 + ... + x_n)^2 = \sum_i x_i^2 + 2 \sum_{i<j} x_i x_j$ we have, as in (i), that

$$E(X_1 + ... + X_n)^2 = \sum_i EX_i^2 + 2 \sum_{i<j} E(X_i X_j). \quad (1)$$

Again from (i), since $E(X_i X_j)$ is finite for each pair (i, j), we have that $E(X_1 + ... + X_n)^2$ is finite.

Using $Var(Z) \geq 0$, we have $(EZ)^2 \leq E(Z^2)$. Therefore $E(X_1 + ... + X_n)$ exists as well.

To get a useful expression for variance of sum of random variables, we assume without loss of generality that $EX_i = 0$ for all i. The first term in the right hand side of equation (1) is then simply the sum of variances of X_i and the second term (without the factor 2) is the sum of covariances.